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New Fixed-Point Theorems for Two Maps and Applications to Problems on Sets with Convex Sections and Minimax Inequalities

K. Q. LAN
Department of Mathematics, Physics and Computer Science
Ryerson University
Toronto, Ontario, Canada M5B 2K3

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Abstract— New fixed-point theorems for two maps defined on product spaces are obtained. These new results only require one of them to satisfy a noncompactness condition. Previous results required each map to satisfy a noncompactness condition. Applications of our results are given to intersection problems for transfer-closed maps, to problems on sets with convex sections, and to minimax inequalities of Sion-type. Our results generalize many well-known results.© 2004 Elsevier Ltd. All rights reserved.

Keywords— Fixed point, Intersection theorem, Sets with convex sections, Minimax inequality

1. INTRODUCTION

Fixed-point problems for a family of multivalued maps defined in product spaces have applications to problems on sets with convex sections, the existence of Nash equilibria in game theory, and minimax inequalities.

The existence of fixed points for a family of multivalued maps defined in product spaces has been studied by several authors. Lan and Webb [1] first obtained fixed-point theorems for a family of multivalued maps defined on product spaces and provided their applications. Each map involved satisfies a noncompactness condition and has an open lower section, that is, the inverse image of every point is open. Lan and Webb's idea was employed by Ansari and Yao [2] to treat the existence of fixed points for a family of multivalued maps defined in product spaces. These maps involved are of different types from those in [1]. Recently, Lan and Wu [3] generalize Lan and Webb's results to the class of maps which possess the local intersection properties. The local intersection properties are more general than the properties of open lower sections and have been widely employed in the fixed-point theorems of Browder type [4, Theorem 7.2, p. 33], continuous selection theorems [5, Theorem 1], and fixed-point theorems for a family of maps defined on product spaces [2]. Some other results also use the local intersection properties, for example, see [6–9].

In this paper, we study fixed-point problems for two maps defined on product spaces. The two maps are required to have the local intersection properties, but only one of them satisfies a

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noncompactness condition. Previous results in [1] and [3] required all of the maps involved to satisfy noncompactness conditions.

We apply our results to problems on sets with convex sections and to inequalities for two functions. We also apply these fixed-point theorems to obtain new intersection results for a multivalued map. These intersection results are well suited to treat variational inequalities for a function and minimax inequalities of Sion-type. Our results generalize many well-known results.

In Section 2, we mention several results obtained in [3]. In Section 3, we give some properties of transfer-lower (or upper)-continuity of functions defined on the product of two spaces. In Section 4, we prove our new fixed-point theorems and an intersection theorem. In Section 5, we apply these fixed-point theorems to treat problems on sets with convex sections and on inequalities for two functions. In the last section, we apply our intersection theorems to variational inequalities for a function and minimax inequalities of Sion-type in a noncompact setting.

2. PRELIMINARIES

In this section, we always assume that X is a nonempty set and Y a topological space. We denote by 2^Y the family of all subsets of Y, and by $\bar{B}$ and B^0 the closure and interior of a subset B of Y, respectively. Let $G : X \to 2^Y$ be a map. We define $G^c, \bar{G}, G^0 = X \to 2^Y$ by $G^c(x) = \{y \in Y : y \notin G(x)\}$, $\bar{G}(x) = \overline{G(x)}$, and $G^0(x) = (G(x))^0$, respectively. We also define maps $G^{-1}, G^* : Y \to 2^X$ by $G^{-1}(y) = \{x \in X : y \in G(x)\}$ and $G^*(y) = \{x \in X : y \notin G(x)\}$, respectively.

The following result was given in [10] (also see Lemma 2.1 in [3]).

LEMMA 2.1. *Let $S, T : X \to Y$ be two maps. Then, the following properties hold.*

(h_1) *For each $x \in X$, $S(x) \subset T(x)$ if and only if $T^*(y) \subset S^*(y)$ for each $y \in Y$.*
(h_2) *$y \notin T(x)$ if and only if $x \in T^*(y)$.*
(h_3) *For each $x \in X$, $(T^*)^*(x) = T(x)$.*
(h_4) *For each $x \in X$, $T(x) \neq \emptyset$ if and only if $\bigcap_{y \in Y} T^*(y) = \emptyset$.*
(h_5) *For each $y \in Y$, $(T^c)^*(y) = T^{-1}(y)$.*
(h_6) *For each $y \in Y$, $(T^{-1})^c(y) = T^*(y)$.*

Recall that a map $F : Y \to 2^X$ is said to have the local intersection property if there exists an open neighborhood $N(y)$ of y such that $\bigcap_{z \in N(y)} F(y) \neq \emptyset$ whenever $F(y) \neq \emptyset$ (see [3] and [5]). It is known that if $F : Y \to 2^X$ has an open lower section, that is, $F^{-1}(x)$ is open in Y for each $x \in X$, then F has the local intersection property. The converse is not true (see Example 1 in [3]). Recall that a map $T : X \to 2^Y$ is said to be transfer-closed if for each $x \in X$ and $y \in T^c(x)$, there exists $x_1 \in X$ such that $y \notin \bar{T}(x_1)$ (see [11]).

We need the following results obtained in [3].

LEMMA 2.2. *The following assertions hold.*

(1) *Let $T : X \to 2^Y$ be a map. Then, T is transfer-closed if and only if $T^* : Y \to 2^X$ has the local intersection property.*
(2) *Let $F : Y \to 2^X$ be a map. Then, F has the local intersection property if and only if F^* is transfer-closed.*

LEMMA 2.3. *Let $F, G : Y \to 2^X$ be maps. Assume that the following conditions hold.*

(a) *$F(y) \subset G(y)$ for each $y \in Y$.*
(b) *$F(y) \neq \emptyset$ for each $y \in Y$.*
(c) *F has the local intersection property.*

Then, G has the local intersection property.

LEMMA 2.4. *Let $F : Y \to 2^X$ be a map. Then, the following assertions are equivalent.*

(i) *F has the local intersection property and $F(y) \neq \emptyset$ for each $y \in Y$.*

(ii) $(\bar{F}^*)^*(y) \neq \emptyset$ *for each* $y \in Y$.

LEMMA 2.5. *Assume that* $F : Y \to 2^X$ *has the local intersection property. Then, the map* $(\bar{F}^*)^* : Y \to 2^X$ *has an open lower section.*

3. TRANSFER-CONTINUITY OF FUNCTIONS DEFINED ON PRODUCT SPACES

In this section, we shall give some properties of transfer lower- (or upper)-continuity of functions defined on $X \times Y$.

Let Z be a topological space. Recall that a function $g : Z \to (-\infty, \infty]$ is said to be lower semicontinuous on Z if the set $\{z \in Z : g(z) > \lambda\}$ is open in Z for each $\lambda \in \mathbb{R}$. $g : Z \to [-\infty, \infty)$ is said to be upper-semicontinuous on Z if the set $\{z \in Z : g(z) < \lambda\}$ is open on Z for each $\lambda \in \mathbb{R}$. It is obvious that g is lower-semicontinuous if and only if $-g$ is upper-semicontinuous.

DEFINITION 3.1. *Let* X *be a set,* Y *a topological space, and* $\lambda \in \mathbb{R}$. *A function* $f : X \times Y \to \mathbb{R}$ *is said to be* λ*-transfer-lower-semicontinuous on* Y *if there exist an open neighborhood* $N(y)$ *of* y *and* $x_1 \in X$ *such that*

$$f(x_1, z) > \lambda, \qquad \text{for each } z \in N(y),$$

whenever $f(x, y) > \lambda$.

f *is said to be transfer-lower-semicontinuous on* Y *if* f *is* λ*-transfer-lower-semicontinuous on* Y *for each* $\lambda \in \mathbb{R}$.

We define maps $F : Y \to 2^X$ *and* $T : X \to 2^Y$ *by*

$$F_\lambda(y) = \{x \in X : f(x, y) > \lambda\} \qquad \text{and} \qquad T_\lambda(x) = \{y \in Y : f(x, y) \leq \lambda\}. \tag{3.1}$$

The following result gives the relations among f, F, and T. Its proof follows from Lemma 2.2 and is omitted.

PROPOSITION 3.1. *Let* $f : X \times Y \to \mathbb{R}$ *be a function. Then, the following are equivalent.*

(i) f *is* λ*-transfer-lower-semicontinuous on* Y.
(ii) *The map* F_λ *defined in (3.1) has the local intersection property.*
(iii) *The map* T_λ *defined in (3.1) is transfer-closed.*

If f is λ-transfer-lower-semicontinuous, it is not clear whether f is β-transfer-lower-semicontinuous for $\beta < \lambda$. The following result provides a sufficient condition for f to be β-transfer-lower-semicontinuous.

PROPOSITION 3.2. *Let* $f : X \times Y \to \mathbb{R}$ *be a function. Assume that the following conditions hold.*

(C_1) f *is* λ*-transfer-lower-semicontinuous on* Y.
(C_2) *For each* $y \in Y$, *there exists* $x_0 \in X$ *such that* $f(x_0, y) > \lambda$.

Then, f *is* β*-transfer-lower-semicontinuous on* Y *for each* $\beta < \lambda$.

PROOF. Let $\beta < \lambda$. We define two maps $F_\lambda, F_\beta : Y \to 2^X$ by

$$F_\lambda(y) = \{x \in X : f(x, y) > \lambda\} \qquad \text{and} \qquad F_\beta(y) = \{x \in X : f(x, y) > \beta\}.$$

Then, $F_\lambda(y) \subset F_\beta(y)$ for each $y \in Y$ since $\beta < \lambda$. By (C_1) and Proposition 3.1 (i) and (ii), F_λ has the local intersection property. By (C_2), $F_\lambda(y) \neq \emptyset$ for each $y \in Y$. It follows from Lemma 2.3 that F_β has the local intersection property. By Proposition 3.1 (i) and (ii), f is β-transfer-lower-semicontinuous on Y. ∎

Closely associated with λ-transfer-lower-semicontinuity is the notion of λ-transfer-upper-semicontinuity.

DEFINITION 3.2. *Let X be a topological space, Y a set, and $\lambda \in \mathbb{R}$. A function $f : X \times Y \to \mathbb{R}$ is said to be λ-transfer-upper-semicontinuous on X if there exist an open neighborhood $N(x)$ of x and $y_1 \in Y$ such that*

$$f(u, y_1) < \lambda, \qquad \text{for each } u \in N(x),$$

whenever $f(x, y) < \lambda$.

f is said to be transfer-upper-semicontinuous on X if f is λ-transfer-upper-semicontinuous on X for each $\lambda \in \mathbb{R}$.

The following result gives the relation between the two semicontinuity maps. Its proof is straightforward and omitted.

LEMMA 3.1. *A function $f : X \times Y \to \mathbb{R}$ is λ-transfer-upper-semicontinuous on X if and only if the function $g : Y \times X \to \mathbb{R}$ defined by $g(y, x) = -f(x, y)$ is $(-\lambda)$-transfer-lower-semicontinuous on X.*

We define maps $G_\lambda : X \to 2^Y$ and $S_\lambda : Y \to 2^X$ by

$$G_\lambda(x) = \{y \in Y : f(x, y) < \lambda\} \qquad \text{and} \qquad S_\lambda(y) = \{x \in X : f(x, y) \geq \lambda\}. \tag{3.2}$$

The following result gives the relations among f, G, and S. Its proof follows from Proposition 3.1 and Lemma 3.1 and is omitted.

PROPOSITION 3.3. *Let $f : X \times Y \to \mathbb{R}$ be a function. Then, the following are equivalent.*

(i) *f is λ-transfer-upper-semicontinuous on X.*
(ii) *The map S_λ defined in (3.2) is transfer-closed.*
(iii) *The map G_λ defined in (3.2) has the local intersection property.*

By Proposition 3.2 and Lemma 3.1, we obtain the following.

PROPOSITION 3.4. *Let $f : X \times Y \to \mathbb{R}$ be a function. Assume that the following conditions hold.*

(i) *f is λ-transfer-upper-semicontinuous on X.*
(ii) *For each $x \in X$, there exists $y_0 \in Y$ such that $f(x, y_0) < \lambda$.*

Then, f is β-transfer-upper-semicontinuous on X for each $\beta > \lambda$.

Now, we give properties of transfer-lower (upper)-semicontinuous functions.

PROPOSITION 3.5. *The following assertions hold.*

(1) *If $f : X \times Y \to \mathbb{R}$ is transfer-lower-semicontinuous on Y, then, the function $h : Y \to (-\infty, \infty]$ defined by $h(y) = \sup_{x \in X} f(x, y)$ is lower-semicontinuous on Y.*
(2) *If $f : X \times Y \to \mathbb{R}$ is transfer-upper-semicontinuous on X, then, the function $g : X \to [-\infty, \infty)$ defined by $g(y) = \inf_{y \in Y} f(x, y)$ is upper-semicontinuous on X.*

PROOF. It is clear that (2) follows from (1) and Lemma 3.1. We only prove (1). Let $\lambda \in \mathbb{R}$ and $y_0 \in \{y \in Y : h(y) > \lambda\}$. Then, there exists $x_0 \in X$ such that $f(x_0, y_0) > \lambda$. Since f is transfer-lower-semicontinuous on Y, there exist $x_1 \in X$ and an open neighborhood $N(y_0)$ of y_0 such that $f(x_1, z) > \lambda$ for $z \in N(y_0)$. This implies $h(z) > \lambda$ for each $z \in N(y_0)$ and the set $\{y \in Y : h(y) > \lambda\}$ is open in Y. Hence, h is lower-semicontinuous on Y and (1) holds. ∎

By Proposition 3.5, we obtain the following new result which will be used in Section 6.

THEOREM 3.1.

(1) *If Y is compact and $f : X \times Y \to \mathbb{R}$ is transfer-lower-semicontinuous on Y, then $\inf_{y \in Y} \sup_{x \in X} f(x, y) = \min_{y \in Y} \sup_{x \in X} f(x, y)$.*
(2) *If X is compact and $f : X \times Y \to \mathbb{R}$ is transfer-upper-semicontinuous on X, then $\sup_{x \in X} \inf_{y \in Y} g(x, y) = \max_{x \in X} \inf_{y \in Y} g(x, y)$.*

4. FIXED-POINT THEOREM FOR TWO MAPS

From now on, we always assume that X and Y are two nonempty convex subsets of Hausdorff topological vector spaces E_1 and E_2, respectively.

We start with the following result on the existence of continuous selections of a multivalued map.

LEMMA 4.1. *Let X_1 be a nonempty compact convex subset of E_1. Assume that $F : X_1 \to 2^Y$ satisfies the following conditions.*

(a) *$F(x) \neq \emptyset$ and is convex for each $x \in X_1$.*

(b) *$F^{-1}(y)$ is open in X_1 for each $y \in Y$.*

Then, there exist $\{y_1, \dots, y_m\} \subset Y$ and a continuous map $g : X_1 \to Y_1$ such that $g(x) \in F(x)$ for each $x \in X_1$, where $Y_1 = \mathrm{co}\{y_1, \dots, y_m\}$.

PROOF. By (a) and (h_4) of Lemma 2.1, we have $X_1 = \bigcup_{y \in Y} F^{-1}(y)$. It follows from compactness of X_1 and (b) that there exists $\{y_1, \dots, y_m\} \subset Y$ such that $X_1 = \bigcup_{j=1}^m F^{-1}(y_j)$. Let $\{f_1, \dots, f_m\}$ be a nonnegative continuous partition of unity subordinate to the open covering $\{F^{-1}(y_1), \dots, F^{-1}(y_m)\}$ of X_1. Then,

$$f_j(x) = 0, \qquad \text{for } x \in X_1 \setminus F^{-1}(y_j), \qquad \text{and} \qquad \sum_{j=1}^{m} f_j(x) = 1, \qquad \text{for } x \in X_1.$$

Let $Y_1 = \mathrm{co}\{y_1, \dots, y_m\}$. Define a map $g : X_1 \to Y_1$ by $g(x) = \sum_{j=1}^m f_j(x) y_j$. Then, g is continuous and $g(x) \in F(x)$ for each $x \in X_1$. ∎

REMARK 4.1. The argument in the proof of Lemma 4.1 belongs to Browder (see the proof of Theorem 1 in [12]). We remark that the range of the continuous section g is contained in a compact convex subset Y_1. We shall use the compactness of Y_1 in Theorem 4.1. There are several generalizations of Lemma 4.1, see, for example, [5,8,13], where the ranges of the continuous selections may not be contained in compact convex subsets.

THEOREM 4.1. *Assume that $T : Y \to 2^X$, and $S : X \to 2^Y$ satisfy the following conditions.*

(1) *For each $y \in Y$, $T(y) \neq \emptyset$ and is convex.*

(2) *For each $x \in X$, $T^{-1}(x)$ is open in Y.*

(3) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D of Y such that for each $y \in Y \setminus D$,*

$$X_0 \cap T(y) \neq \emptyset.$$

(4) *For each $x \in X$, $S(x) \neq \emptyset$ and is convex.*

(5) *For each $y \in Y$, $S^{-1}(y)$ is open in X.*

Then, there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in T(y_0)$ and $y_0 \in S(x_0)$.

PROOF. We first prove the following assertion.

(H) There exists a nonempty compact convex subset X_1 of X such that

$$Y = \bigcup_{x \in X_1} T^{-1}(x). \tag{4.1}$$

In fact, by (1) and (h_4) of Lemma 2.1, we have

$$\bigcup_{x \in X} T^{-1}(x) = Y. \tag{4.2}$$

We consider three cases.

(i) If X is compact, let $X_1 = X$. Then, (4.2) becomes (4.1).

(ii) If Y is compact, by (4.2) and (2), there exists $\{x_1, \dots, x_n\} \subset X$ such that $Y = \bigcup_{i=1}^n T^{-1}(x_i)$. Hence, (4.1) holds with $X_1 = \mathrm{co}\{x_1, \dots, x_n\}$.

(iii) If X and Y are not compact, we have $Y \setminus \bigcup_{x \in X_0} T^{-1}(x) \subset D$ by (3).

Since $D \subset Y$ is compact, it follows from (4.2) that there exist $\{x_1, \dots, x_n\} \subset X$ such that $D \subset \bigcup_{i=1}^n T^{-1}(x_i)$. This implies $Y \setminus \bigcup_{x \in X_0} T^{-1}(x) \subset \bigcup_{i=1}^n T^{-1}(x_i)$ and $Y = \bigcup_{x \in X_0} T^{-1}(x) \cup (\bigcup_{i=1}^n T^{-1}(x_i))$. Hence, (4.1) holds with $X_1 = \mathrm{co}\{X_0 \cup \{x_1, \dots, x_n\}\}$.

By (4.1) and (h_4) of Lemma 2.1, we obtain $T(y) \cap X_1 \neq \emptyset$ for each $y \in Y$. We define a map $F : X_1 \to 2^Y$ by $F(x) = S(x)$. Note that $F^{-1}(y) = S^{-1}(y) \cap X_1$ for each $y \in Y$. It is easy to verify that F satisfies all the conditions of Lemma 4.1. Therefore, there exist a nonempty compact convex subset Y_1 of Y and a continuous map $g : X_1 \to Y_1$ such that

$$g(x) \in F(x) = S(x), \qquad \text{for each } x \in X_1. \tag{4.3}$$

We define $T_1 : Y_1 \to 2^{X_1}$ by $T_1(y) = T(y) \cap X_1$. Then, T_1 satisfies the following conditions.

(a) $T_1(y) \neq \emptyset$ and is convex for each $y \in Y_1$.

(b) $T_1^{-1}(x)$ is open in Y_1 for each $x \in X_1$ since $T_1^{-1}(x) = T^{-1}(x) \cap Y_1$ and $T^{-1}(x)$ is open in Y.

By Lemma 4.1, there exists a continuous map $h : Y_1 \to X_1$ such that

$$h(x) \in T_1(x) \subset T(x), \qquad \text{for each } x \in Y_1. \tag{4.4}$$

Since X_1 is a compact convex subset of X and the composite map $h(g) : X_1 \to X_1$ is continuous, it follows from Tychonoff's fixed-point theorem [14] that there exists $x_0 \in X_1$ such that $x_0 = h(g(x_0))$. Let $y_0 = g(x_0)$. By (4.4) and (4.3), we have $x_0 \in T(y_0)$ and $y_0 \in S(x_0)$. ∎

Theorem 4.1 generalizes Theorem 2.1 with $I = \{1, 2\}$ in [1], where S satisfies an extra compactness condition and T satisfies condition (3) of Theorem 4.1 even when X is compact.

The following result is a slight generalization of Theorem 4.1 and generalizes Theorem 2.2 with $I = \{1, 2\}$ in [1].

THEOREM 4.2. *Assume that $A : Y \to 2^X$ and $B : X \to 2^Y$ satisfy the following conditions.*

(i) *For each $y \in Y$, $A(y) \neq \emptyset$.*

(ii) *For each $x \in X$, $A^{-1}(x)$ is open in Y.*

(iii) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D of Y such that for each $y \in Y \setminus D$,*

$$X_0 \cap \mathrm{co}\, A(y) \neq \emptyset.$$

(iv) *For each $x \in X$, $B(x) \neq \emptyset$.*

(v) *For each $y \in Y$, $B^{-1}(y)$ is open in X.*

Then, there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in \mathrm{co}\, A(y_0)$ and $y_0 \in \mathrm{co}\, B(x_0)$.

PROOF. Define a map $T : Y \to 2^X$ by $T(y) = \mathrm{co}\, A(y)$ and a map $S : X \to 2^Y$ by $S(x) = \mathrm{co}\, B(x)$. Then, it is easy to verify that T and S satisfy (1), (3), and (4) of Theorem 4.1. By Lemma 5.1 in [13], (ii) and (v) imply that T and S satisfy (2) and (5) of Theorem 4.1. The result follows from Theorem 4.1. ∎

The following result generalizes Theorem 4.2 to maps which have the local intersections properties.

THEOREM 4.3. *Let $\gamma_1, \psi_1 : Y \to 2^X$ and $\gamma_2, \psi_2 : X \to 2^Y$ be maps. Assume that the following conditions hold.*

(H_1) *For each $y \in Y$, $\gamma_1(y) \subset \psi_1(y)$.*

(H_2) *For each $y \in Y$, $\gamma_1(y) \neq \emptyset$.*

(H_3) *γ_1 has the local intersection property.*

(H_4) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D of Y such that for each $y \in Y \setminus D$,*

$$X_0 \cap \mathrm{co}\left(\bar{\psi}_1^*\right)^*(y) \neq \emptyset.$$

(H_5) *For each* $x \in X$, $\gamma_2(x) \subset \psi_2(x)$.
(H_6) *For each* $x \in X$, $\gamma_2(x) \neq \emptyset$.
(H_7) γ_2 *has the local intersection property.*

Then, there exists $(x_0, y_0) \in X \times Y$ *such that* $x_0 \in \operatorname{co}\psi_1(y_0)$ *and* $y_0 \in \operatorname{co}\psi_2(x_0)$.

PROOF. We define a map $A : Y \to 2^X$ by $A(y) = (\bar{\psi}_1^*)^*(y)$ and a map $B : X \to 2^Y$ by $B(x) = (\bar{\psi}_2^*)^*(x)$. By (H_1)–(H_3), (H_5)-(H_7), and Lemma 2.3, ψ_1 and ψ_2 have the local intersection properties. Moreover, $\psi_1(y) \neq \emptyset$ for each $y \in Y$ and $\psi_2(x) \neq \emptyset$ for each $x \in X$. It follows from Lemma 2.4 that $A(y) = (\bar{\psi}_1^*)^*(y) \neq \emptyset$ for each $y \in Y$ and $B(x) = (\bar{\psi}_2^*)^*(x) \neq \emptyset$ for each $x \in X$. Hence, (i) and (iv) of Theorem 4.2 hold. Since ψ_1 and ψ_2 have the local intersection properties, it follows from Lemma 2.5 that $\phi_1^{-1}(x)$ is open in Y for each $x \in X$ and $\phi_2^{-1}(y)$ is open in X for each $y \in Y$. Hence, (ii) and (v) of Theorem 4.2 hold. It is clear that (H_4) implies (iii) of Theorem 4.2. It follows from Theorem 4.2 that there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in \operatorname{co} A(y_0) = \operatorname{co}(\bar{\psi}_1^*)^*(y_0)$ and $y_0 \in \operatorname{co} B(x_0) = \operatorname{co}(\bar{\psi}_2^*)^*(x_0)$. Since $(\bar{\psi}_1^*)^*(y_0) \subset \psi_1(y_0)$ and $(\bar{\psi}_2^*)^*(x_0) \subset \psi_2(x_0)$, we have $x_0 \in \operatorname{co}\psi_1(y_0)$ and $y_0 \in \operatorname{co}\psi_2(x_0)$. ∎

Theorem 4.3 generalizes Theorem 3.1 in [3] with $I = \{1, 2\}$, where ϕ_2 satisfies an extra condition and ϕ_1 satisfies (H_4) of Theorem 4.3 even when X is compact.

By Theorem 4.3, we obtain the following new intersection theorem which will be applied to obtain new variational inequalities for functions defined on products in Section 6.

THEOREM 4.4. *Let* $S, S_1, T, T_1 : X \to 2^Y$ *be four maps. Assume that the following conditions hold.*

(P_1) *For each* $x \in X$, $T_1(x) \subset T(x) \subset S_1(x) \subset S(x)$.
(P_2) S *is transfer-closed.*
(P_3) *For each* $y \in Y$, $S_1^*(y)$ *is convex.*
(P_4) *If* X *and* Y *are not compact, assume that there exist a nonempty compact convex subset* X_0 *of* X *and a nonempty compact subset* D_Y *of* Y *such that for each* $y \in Y \setminus D_Y$,

$$X_0 \cap \operatorname{co}\left(\bar{S}_1\right)^*(y) \neq \emptyset.$$

(P_5) *For each* $x \in X$, $T_1(x) \neq \emptyset$.
(P_6) T_1 *has the local intersection property.*
(P_7) *For each* $x \in X$, $T(x)$ *is convex.*

Then, $\bigcap_{x\in X} S(x) \neq \emptyset$.

PROOF. Assume that the result is not true. Then, $\bigcap_{x\in X} S(x) = \emptyset$. By ($h_4$) of Lemma 2.1, we have $S^*(y) \neq \emptyset$ for each $y \in Y$. Let $\gamma_1 = S^*$, $\psi_1 = S_1^*$, $\gamma_2 = T_1$, and $\psi_2 = T$. Then, $\gamma_1, \gamma_2, \psi_1$, and ψ_2 satisfy all the conditions of Theorem 4.3. It follows from Theorem 4.3 that there exists $(x, y) \in X \times Y$ such g $x \in \operatorname{co} S_1^*(y)$ and $y \in \operatorname{co} T(x)$. It follows from ($P_3$) and ($P_7$) that $x \in S_1^*(y)$ and $y \in T(x)$. Hence, we obtain $y \notin S_1(x)$ and $y \in T(x)$. Since $T(x) \subset S_1(x)$ for each $x \in X$, we have $y \notin T(x)$ and $y \in T(x)$, a contradiction. Hence, $\bigcap_{x\in X} S(x) \neq \emptyset$. ∎

5. PROBLEMS ON SETS WITH CONVEX SECTIONS

In this section, we apply Theorem 4.3 to problems on sets with convex sections.

THEOREM 5.1. *Let* A_1, A_2, B_1, *and* B_2 *be four subsets of* $X \times Y$. *Assume that the following conditions hold.*

(i) $A_1 \subset B_1$.
(ii) *For each* $y \in Y$, $\{x \in X : (x, y) \in A_1\} \neq \emptyset$.
(iii) *For each* $(x, y) \in A_1$, *there exists an open neighbourhood* $N(y)$ *of* y *and* $x_1 \in X$ *such that* $\{x_1\} \times N(y) \subset A_1$.

(iv) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D of Y such that for each $y \in Y \setminus D$,*

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : (y,z) \notin B_1\}}\right\} \neq \emptyset.$$

(v) $A_2 \subset B_2$.
(vi) *For each $x \in X$, $\{y \in Y : (x,y) \in A_2\} \neq \emptyset$.*
(vii) *For each $(x,y) \in A_2$, there exist an open neighborhood $N(x)$ of x and $y_1 \in Y$ such that $N(x) \times \{y_1\} \subset A_2$.*

Then, there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in \operatorname{co}\{x \in X : (x, y_0) \in B_1\}$ and $y_0 \in \operatorname{co}\{y \in Y : (x_0, y) \in B_2\}$. Moreover, if C_1, C_2 are subsets of $X \times Y$ and satisfy $\operatorname{co} B_1 \subset C_1$ and $\operatorname{co} B_2 \subset C_2$, then $C_1 \cap C_2 \neq \emptyset$.

PROOF. We define $\gamma_1, \phi_1 : Y \to 2^X$ by

$$\gamma_1(y) = \{x \in X : (x,y) \in A_1\} \qquad \text{and} \qquad \phi_1(y) = \{x \in X : (x,y) \in B_1\},$$

and $\gamma_2, \phi_2 : X \to 2^Y$ by

$$\gamma_2(x) = \{y \in Y : (x,y) \in A_2\} \qquad \text{and} \qquad \phi_2(x) = \{y \in Y : (x,y) \in B_2\}.$$

Then, γ_1, γ_2, ϕ_1, and ϕ_2 satisfy all the conditions of Theorem 4.3. The first result follows from Theorem 4.3 and the second follows from the first. ∎

REMARK 5.1. The first result of Theorem 5.1 generalizes Theorem 3.2 with $I = \{1,2\}$ in [3], where B_2 is required to satisfy an extra condition, and Theorem 2.3 in [1]. The second result of Theorem 5.1 generalizes Theorem 3.3 with $I = \{1,2\}$ in [3], Theorem 3.2 in [1], Theorems 15 and 16 in [15], and Theorem 2 in [16].

Let Z be a convex subset of a topological space and $\lambda \in \mathbb{R}$. Recall that a function $g : Z \to \mathbb{R}$ is said to be λ-quasiconcave if the set $\{z \in Z : g(z) > \lambda\}$ is convex, g is said to be λ-quasiconvex if $-g$ is λ-quasiconcave. $g : Z \to \mathbb{R}$ is said to be quasiconcave if g is λ-quasiconvex for each $\lambda \in \mathbb{R}$. g is said to be quasiconvex if $-g$ is quasiconcave.

We give an analytical formulation of Theorem 5.1.

THEOREM 5.2. *Let $f_1, f_2, g_1, g_2, h_1, h_2 : X \times Y \to \mathbb{R}$ be functions and let $t_1, t_2 \in \mathbb{R}$. Assume that the following conditions hold.*

(a) $f_i(x,y) \le g_i(x,y) \le h_i(x,y)$ *for each $i \in I = \{1,2\}$ and each $(x,y) \in X \times Y$.*
(b) *There exists $\lambda_1 \ge t_1$ such that for each $y \in Y$, there exists $x \in X$ such that $f_1(x,y) > \lambda_1$.*
(c) *f_1 is λ_1-transfer-lower-semicontinuous on Y.*
(d) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D of Y such that for each $y \in Y \setminus D$,*

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : g_1(x,z) \le \lambda_1\}}\right\} \neq \emptyset.$$

(e) *There exists $\lambda_2 \ge t_2$ such that for each $x \in X$, there exists $y \in Y$ such that $f_2(x,y) > \lambda_2$.*
(f) *f_2 is λ_2-transfer-lower-semicontinuous on X.*
(g) *For each $y \in Y$, $h_1(\cdot, y)$ is quasiconcave on X.*
(h) *For each $x \in X$, $h_2(x, \cdot)$ is quasiconcave on Y.*

Then, there exists $(x_0, y_0) \in X \times Y$ such that

$$h_1(x_0,y_0) > t_1 \qquad \text{and} \qquad h_2(x_0,y_0) > t_2.$$

PROOF. We define $\gamma_1, \phi_1 : Y \to 2^X$ by

$$\gamma_1(y) = \{x \in X : f_1(x,y) > t_1\} \qquad \text{and} \qquad \phi_1(y) = \{x \in X : g_1(x,y) > t_1\},$$

and $\gamma_2, \phi_2 : X \to 2^Y$ by

$$\gamma_2(x) = \{y \in Y : f_2(x,y) > t_2\} \qquad \text{and} \qquad \phi_2(x) = \{y \in Y : g_2(x,y) > t_2\}.$$

Since $\lambda_1 \geq t_1$, it follows from Conditions (b), (c), and Propositions 3.2 and 3.1 that γ_1 has the local intersection property. Similarly, γ_2 has the local intersection property. Since $\lambda_1 \geq t_1$, (d) implies (H_4) of Theorem 4.3. It is easy to verify that (H_1), (H_2), (H_5), and (H_6) of Theorem 4.3 hold. It follows from Theorem 4.3 that there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in \operatorname{co} \phi_1(y_0)$ and $y_0 \in \operatorname{co} \phi_2(x_0)$. Since $g_i(x_0, y_0) \leq h_i(x_0, y_0)$ and h_i is quasiconcave for each $i = 1, 2$, $h_i(x_0, y_0) > t_i$ for $i = 1, 2$. ∎

REMARK 5.2. Theorem 5.2 generalizes Theorem 3.4 with $I = \{1, 2\}$ in [3]. Even when $f_i = g_i = h_i$, Theorem 5.2 generalizes Theorem 2.5 in [1] and Theorem 3 in [17].

6. MINIMAX INEQUALITIES

In this section, we apply Theorem 4.4 to obtain some new variational inequalities and generalize Sion's minimax inequalities to a noncompact setting.

THEOREM 6.1. *Let $f, f_1, g, g_1 : X \times Y \to \mathbb{R}$ be four functions and let $\lambda \in \mathbb{R}$. Assume that the following conditions hold.*

(V_1) *$f(x,y) \leq f_1(x,y) \leq g(x,y) \leq g_1(x,y)$ for each $(x,y) \in X \times Y$.*
(V_2) *f is λ-transfer-lower-semicontinuous on Y.*
(V_3) *For each $y \in Y$, $f_1(\cdot, y)$ is λ-quasiconcave on X.*
(V_4) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D_Y of Y such that for each $y \in Y \setminus D_Y$,*

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : f_1(x,z) \leq \lambda\}}\right\} \neq \emptyset.$$

(V_5) *There exists $\lambda_1 \leq \lambda$ such that for each $x \in X$, there exists $y \in Y$ such that $g_1(x,y) < \lambda_1$.*
(V_6) *g_1 is λ_1-transfer-upper-semicontinuous on X.*
(V_7) *For each $x \in X$, $g(x, \cdot)$ is λ-quasiconvex on Y.*

Then, there exists $y_0 \in Y$ such that $f(x, y_0) \leq \lambda$ for all $x \in X$.

PROOF. We define maps $S, S_1, T, T_1 : X \to 2^Y$ by $S(x) = \{y \in Y : f(x,y) \leq \lambda\}$, $S_1(x) = \{y \in Y : f_1(x,y) \leq \lambda\}$, $T(x) = \{y \in Y : g(x,y) < \lambda\}$, and $T_1(x) = \{y \in Y : g_1(x,y) < \lambda\}$. By ($V_5$), ($V_6$), and Propositions 3.3 and 3.4, T_1 satisfies (P_5) and (P_6) of Theorem 4.4. It is easy to verify that S, S_1, T, T_1 satisfy all the other conditions of Theorem 4.4. The result follows from Theorem 4.4. ∎

In Theorem 6.1, if $f = f_1 = g = g_1$, we obtain the following.

COROLLARY 6.1. *Let $f : X \times Y \to \mathbb{R}$ be a function and let $\lambda \in \mathbb{R}$. Assume that the following conditions hold.*

(i) *f is λ-transfer-lower-semicontinuous on Y.*
(ii) *For each $y \in Y$, $f(\cdot, y)$ is λ-quasiconcave on X.*
(iii) *If X and Y are not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D_Y of Y such that for each $y \in Y \setminus D_Y$,*

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : f(x,z) \leq \lambda\}}\right\} \neq \emptyset.$$

(iv) *There exists $\lambda_1 \leq \lambda$ such that for each $x \in X$, there exists $y \in Y$ such that $f(x,y) < \lambda_1$.*
(v) *f is λ_1-transfer-upper-semicontinuous on X.*
(vi) *For each $x \in X$, $f(x, \cdot)$ s λ-quasiconvex on Y.*

Then, there exists $y_0 \in Y$ such that $f(x, y_0) \leq \lambda$ for all $x \in X$.

NOTATION. Let $\lambda_f = \sup_{x \in X} \inf_{y \in Y} f(x,y)$.

By Theorem 6.1, we obtain the following new inequalities for functions defined on $X \times Y$.

THEOREM 6.2. *Let $f, f_1, g, g_1 : X \times Y \to \mathbb{R}$ be four functions. Assume that the following conditions hold.*

(v_0) $\lambda_{g_1} \in [-\infty, \infty)$.
(v_1) $f(x,y) \le f_1(x,y) \le g(x,y) \le g_1(x,y)$, *for each* $(x,y) \in X \times Y$.
(v_2) f *is transfer-lower-semicontinuous on* Y.
(v_3) *For each* $y \in Y$, $f_1(\cdot, y)$ *is quasiconcave on* X.
(v_4) *If* X *and* Y *are not compact, assume that there exist* $\lambda_1 \in (\lambda_{g_1}, \infty)$, *a nonempty compact convex subset* X_0 *of* X, *and a nonempty compact subset* D_Y *of* Y *such that for each* $y \in Y \setminus D_Y$,

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : f_1(x,z) \le \lambda_1\}}\right\} \ne \emptyset.$$

(v_6) g_1 *is* λ*-transfer-upper-semicontinuous on* X *for each* $\lambda \in (\lambda_{g_1}, \lambda_1]$.
(v_7) *For each* $x \in X$, $g(x, \cdot)$ *is* λ*-quasiconvex on* Y *for each* $\lambda \in (\lambda_{g_1}, \lambda_1]$.

Then, we have the following inequalities:

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \le \sup_{x\in X} \inf_{y\in Y} g_1(x,y) \le \inf_{y\in Y} \sup_{x\in X} g_1(x,y).$$

PROOF. Let $\lambda \in (\lambda_{g_1}, \lambda_1]$. For each $x \in X$, there exists $y \in Y$ such that $g_1(x,y) < \lambda$. Hence, (V_5) of Theorem 6.1 holds. Since $\lambda \le \lambda_1$, (v_4) implies (V_4). It is obvious that (v_j) implies (V_j) for each $j \in \{1,2,3,6,7\}$. It follows from Theorem 6.1 that there exists $y_0 \in Y$ such that $f(x,y_0) \le \lambda$, for all $x \in X$. This implies $\inf_{y\in Y} \sup_{x\in X} f(x,y) \le \lambda$ for each $\lambda \in (\lambda_{g_1}, \lambda_1)$. Hence, $\inf_{y\in Y} \sup_{x\in X} f(x,y) \le \lambda_{g_1}$. Hence, the first inequality holds. It is clear that the second inequality holds. ∎

As a special case of Theorem 6.2, we obtain a new minimax inequality.

COROLLARY 6.2. *Assume that* $f : X \times Y \to \mathbb{R}$ *satisfies the following conditions.*

(a) $\lambda_f \in [-\infty, \infty)$.
(b) f *is transfer-lower-semicontinuous on* Y.
(c) *For each* $y \in Y$, $f(\cdot, y)$ *is quasiconcave on* X.
(d) *If* X *and* Y *are not compact, assume that there exist* $\lambda_1 \in (\lambda_f, \infty)$, *a nonempty compact convex subset* X_0 *of* X, *and nonempty compact subset* D_Y *of* Y *such that for each* $y \in Y \setminus D_Y$,

$$X_0 \cap \operatorname{co}\left\{x \in X : y \notin \overline{\{z \in Y : f(x,z) \le \lambda_1\}}\right\} \ne \emptyset.$$

(e) f *is* λ*-transfer-upper-semicontinuous on* X, *for each* $\lambda \in (\lambda_f, \lambda_1]$.
(f) *For each* $x \in X$, $f(x, \cdot)$ *is quasiconvex on* Y, *for each* $\lambda \in (\lambda_f, \lambda_1]$.

Then, $\inf_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y)$.

By Corollary 6.2 and Theorem 3.1, we obtain the following.

COROLLARY 6.3. *Let* X *and* Y *be two nonempty compact convex subsets of* E *and* F. *Assume that* $f : X \times Y \to \mathbb{R}$ *satisfies (a)–(c), (e), and (f) of Corollary 6.2. Then,* $\min_{y\in Y} \sup_{x\in X} f(x,y) = \max_{x\in X} \inf_{y\in Y} f(x,y)$.

Corollary 6.3 generalizes Sion's minimax inequality (see Theorem 3.4 in [18]), Theorem 5 in [17], and Theorem 16 in [12]) by relaxing the lower semicontinuity and upper semicontinuity of f on Y and X.

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